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Computing Invariant Measures of Piecewise Convex Transformations[1]

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Let $S: [0, 1] \to [0, 1]$ be a piecewise convex transformation satisfying some conditions which guarantee the existence of an absolutely continuous invariant probability measure. We prove the convergence of a class of Markov finite approximations for computing the invariant measure, using a compactness argument for L^1-spaces.

KEY WORDS: Invariant measure; Frobenius–Perron operator.

1. INTRODUCTION

Exploring statistical properties of dynamical systems was originally motivated by the examination of the Boltzmann ergodic hypothesis in statistical physics, and this led to the investigation of measure-preserving transformations.[3,17] In physical sciences, many problems are closely related to the existence and computation of the density of an absolutely continuous invariant measure for nonsingular transformations on measure spaces.[10] For example, in neural networks, condensed matter physics, turbulence in fluid flow, arrays of Josephson junctions, large-scale laser arrays, reaction-diffusion systems, etc., "coupled map lattaces" appear as models for phase transitions, in which the evolution and convergence of densities under the action of the so-called Frobenius–Perron operator are examined. Thus, from the physical point of view, the computation of the invariant density is very important.

The study of one-dimensional dynamics constitutes a basis for studying general dynamical systems.[14] For one-dimensional mappings which are

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piecewise C^2 and stretching or piecewise C^2 and convex with a strong repellor, the existence of an invariant probability measure which is absolutely continuous with respect to the Lebesgue measure has been proved by Lasota and Yorke.[11,12]

For the computation of one-dimensional absolutely continuous invariant measures for the class of piecewise C^2 and stretching mappings of $[0, 1]$, Li[13] proved the convergence of Ulam's piecewise constant approximation method.[16] Some higher order methods have been developed in refs. 1, 5, 6, and 9. A unified approach was proposed in ref. 4. Error estimates of these methods were obtained in refs. 2, 4, and 8.

Recently the convergence of Ulam's method was proved for the class of piecewise convex mappings with a strong repellor by Miller.[15] A key point in his proof is a simple observation that Ulam's piecewise constant approximations of a decreasing function are decreasing functions. In this paper, we explore some new properties of higher order Markov finite approximations and establish their convergence after we show that such approximations also preserve the monotonicity of the function, using the fact that any subset of monotonic functions in $L^1(0, 1)$ that is uniformly bounded in L^∞-norm must be precompact in $L^1(0, 1)$.

After giving some preliminaries in the next section, we prove the convergence of the first-order Markov finite approximation method in Section 3. Section 4 is devoted to the convergence of the second-order method. Some numerical results are given in Section 5. We conclude in Section 6.

2. PIECEWISE CONVEX TRANSFORMATIONS

Let $S: [0, 1] \to [0, 1]$ be a measurable transformation such that $m(A) = 0$ implies $m(S^{-1}(A)) = 0$ for every Lebesgue measurable subset of $[0, 1]$, where m denotes the Lebesgue measure. The operator $P: L^1(0, 1) \to L^1(0, 1)$ defined by

$$\int_A Pf\, dm = \int_{S^{-1}(A)} f\, dm \tag{1}$$

for every measurable $A \subset [0, 1]$ is called the Frobenius–Perron operator associated with S. It is well known[10] that for $f \geqslant 0$ and $\|f\| \equiv \int_0^1 |f|\, dm = 1$, the absolutely continuous probability measure

$$\mu(A) = \int_A f\, dm \quad \forall \text{ measurable sets } A \subset [0, 1]$$

is invariant under S if and only if f is a fixed point of P, i.e., $Pf = f$. We call f the density of μ.

A basic and simple property of P which is useful in this paper is given below without proof. For more detailed discussion of P, see ref. 10.

Proposition 2.1. P is a Markov operator. That is, $Pf \geqslant 0$ and $\|Pf\| = \|f\|$ if $f \geqslant 0$.

Now we state the existence theorem for a class of mappings that are piecewise convex with a strong repellor, the proof of which is referred to ref. 10 or 12.

Theorem 2.1. Let $S: [0, 1] \to [0, 1]$ satisfy the following conditions:

(i) There is a partition $0 = a_0 < a_1 < \cdots < a_r = 1$ of $[0, 1]$ such that $S|_{[a_{i-1}, a_i)}$ is of C^2 for each $i = 1,..., r$.

(ii) $S'(x) > 0$ and $S''(x) \geqslant 0$ for all $x \in [0, 1)$, where $S'(a_i)$ and $S''(a_i)$ are right derivatives.

(iii) $S(a_i) = 0$ for each integer $i = 1,..., r$.

(iv) $S'(0) > 1$.

Then there exists a unique absolutely continuous invariant probability measure with density f^*. Moreover, f^* is a decreasing function, and $\{P^n\}$ is asymptotically stable in the sense that $\lim_{n \to \infty} P^n f = f^*$ for every density $f \in L^1(0, 1)$.

Remark 2.1. The point $x = 0$ is called a strong repellor since the trajectory $\{S(x_0), S^2(x_0),...\}$, starting from a point $x_0 \in [0, a_1)$, will eventually leave $[0, a_1)$. This property is essential in the proof of asymptotic stability of $\{P^n\}$.

Remark 2.2. It was shown in ref. 10, Theorem 6.3.1, that for the piecewise convex transformation S with a strong repellor, the Frobenius–Perron operator P leaves the set of nonnegative decreasing functions invariant, and for any decreasing density $f \in L^1(0, 1)$,

$$Pf(x) \leqslant \frac{1}{\lambda} f(0) + K \tag{2}$$

where $\lambda = S'(0) > 1$ and $K = \sum_{i=2}^{r} 1/(a_{i-1} S'(a_{i-1}))$. Thus, we have

$$P^n f(x) \leqslant f(0) + \frac{\lambda K}{\lambda - 1}, \qquad \forall n$$

which guarantees the existence of f^*.

Since P is an infinite-dimensional Markov operator, it is natural to construct its finite approximations, which are also Markov operators.

Ulam's original piecewise constant method preserves the Markov property of P. Moreover, it preserves the monotonicity property of functions, which leads to the convergence of Ulam's method for piecewise convex transformations with a strong repellor.[15] In the following two sections, we show that higher order Markov approximation methods that were developed in ref. 5 for piecewise stretching mappings also preserve the monotonicity of functions. The basic idea in the proof is that any convex combination of several real numbers is between the minimal one and the maximal one. With the help of the inequality (2), we can prove that these methods converge with a higher convergence rate.

3. THE FIRST-ORDER MARKOV FINITE APPROXIMATION

In next two sections we consider a class of finite approximations of P which are Markov operators of finite dimensions. Ulam's piecewise constant approximations are Markov finite approximations in the above sense as well as a projection method. But a higher order projection method[6] is not a Markov approximation. Here we analyze the convergence of the piecewise linear Markov approximation scheme for computing the fixed point of P when S is piecewise convex with a strong repellor.

Divide the interval $[0, 1]$ into n equal subintervals $I_i = [x_{i-1}, x_i]$ with the length $h = 1/n$. Then the corresponding continuous piecewise-linear finite element space T_n is $(n+1)$-dimensional. Its standard basis consists of the *tent functions*

$$e_i(x) = e\left(\frac{x - x_i}{h}\right), \qquad i = 0, 1,..., n$$

where

$$e(x) = (1 - |x|)\,\chi_{[-1,1]}(x), \qquad -\infty < x < \infty$$

Here χ_A represents the characteristic function of A. This basis has the property that $f = \sum_{i=0}^{n} f_i e_i$ if and only if $f(x_i) = f_i$ for all i. In particular,

$$\sum_{i=0}^{n} e_i(x) \equiv 1, \qquad x \in [0, 1]$$

Note that the support of e_i is $I_i \cup I_{i+1}$ for $i = 1, 2,..., n-1$, and those of e_0 and e_n are I_1 and I_n, respectively. In the following, denote

$$f_i = \frac{1}{h}\int_{I_i} f\, dm$$

which is the average value of f over the ith subinterval I_i. Now define

$$Q_n f = f_1 e_0 + \sum_{i=1}^{n-1} \frac{f_i + f_{i+1}}{2} e_i + f_n e_n \tag{3}$$

It was proved in ref. 5 that Q_n: $L^1(0, 1) \to L^1(0, 1)$ is a Markov operator of finite rank, $\lim_{n\to\infty} Q_n f = f$ strongly for all $f \in L^1(0, 1)$, and $\bigvee_0^1 Q_n f \leqslant \bigvee_0^1 f$. It is also easy to see that $\| Q_n f \|_\infty \leqslant \| f \|_\infty$ for all $f \in L^\infty(0, 1)$. From Remark 2.2, the Frobenius–Perron operator P corresponding to the piecewise convex transformation S maps the set of nonnegative decreasing functions into itself. The following result indicates that Q_n has the same property.

Lemma 3.1. If $f \in L^1(0, 1)$ is decreasing, then $Q_n f$ is also decreasing.

Proof. Let $0 \leqslant x \leqslant y \leqslant 1$ and $x \in I_i$ for some $i = 1, 2, ..., n$. First suppose $1 < i < n$ and $y \in I_i$. Then

$$Q_n f(x) = \frac{f_{i-1} + f_i}{2} e_{i-1}(x) + \frac{f_i + f_{i+1}}{2} e_i(x)$$

$$Q_n f(y) = \frac{f_{i-1} + f_i}{2} e_{i-1}(y) + \frac{f_i + f_{i+1}}{2} e_i(y)$$

Since f is decreasing, $f_{i-1} \geqslant f_{i+1}$. Noting that $e_{i-1}(x) = 1 - e_i(x) \geqslant e_{i-1}(y) = 1 - e_i(y)$, we have

$$\begin{aligned}
&Q_n(x) - Q_n(y)\\
&= \frac{f_{i-1} + f_i}{2} [e_{i-1}(x) - e_{i-1}(y)] + \frac{f_i + f_{i+1}}{2} [e_i(x) - e_i(y)]\\
&= \frac{f_{i-1} + f_i}{2} [e_{i-1}(x) - e_{i-1}(y)] - \frac{f_i + f_{i+1}}{2} [e_{i-1}(x) - e_{i-1}(y)]\\
&= \frac{f_{i-1} - f_{i+1}}{2} [e_{i-1}(x) - e_{i-1}(y)] \geqslant 0
\end{aligned}$$

The case $i = 1$ or $i = n$ can be proved similarly. Now suppose $y \in I_j$ with $1 < i < j < n$. Then

$$Q_n f(y) = \frac{f_{j-1} + f_j}{2} e_{j-1}(x) + \frac{f_j + f_{j+1}}{2} e_j(x)$$

Since f is decreasing, we have

$$f_{i-1} \geqslant f_i \geqslant f_{j-1} \geqslant f_j$$

From $e_{i-1}(x) + e_i(x) = 1$ and $e_{j-1}(y) + e_j(y) = 1$,

$$\begin{aligned} Q_n f(x) &= (1 - e_i(x)) f_{i-1} + e_i(x) f_i \geqslant f_i \geqslant f_{j-1} \\ &\geqslant (1 - e_j(y)) f_{j-1} + e_j(y) f_j = Q_n f(y) \end{aligned}$$

By the same argument, one can prove the same result in the case $i = 1$ or $j = n$. ∎

Let P_n be the restriction of $Q_n P$ on the finite element space T_n. Then P_n is a Markov operator of finite rank, and $\lim_{n \to \infty} P_n f = Pf$ strongly for any $f \in L^1(0, 1)$. The representation of P_n under any density function basis of T_n is given by an $(n+1) \times (n+1)$ stochastic matrix. From the Frobenius–Perron theory of nonnegative matrices, there is a fixed density f_n of P_n in T_n (see also ref. 5 for a proof). The following lemma shows that f_n can actually be taken to be decreasing.

Lemma 3.2. P_n: $T_n \to T_n$ has a continuous piecewise-linear fixed density function $f_n \in D_n$ where

$$D_n = \{ f \in T_n \mid f \geqslant 0, \ \|f\| = 1, f \text{ is decreasing} \}$$

Proof. Since both Q_n and P keep the set of decreasing nonnegative functions invariant, so does P_n. Thus P_n maps D_n into itself, since $\|P_n f\| = \|f\|$ for $f \geqslant 0$. It is easy to see that D_n is a compact convex subset of T_n. So the assertion follows from the Brouwer fixed-point theorem. ∎

Now we can prove the convergence of the piecewise linear Markov approximation method.

Theorem 3.1. Suppose S: $[0, 1] \to [0, 1]$ satisfies the conditions of Theorem 2.1. Let $f_n \in D_n$ be a sequence of continuous decreasing piecewise-linear fixed density functions of P_n. Then f_n converge to the unique fixed density f^* of P in $L^1(0, 1)$.

Proof. First we show that the sequence of nonnegative numbers $f_n(0)$ is bounded above. In fact, from (2), we have for $x \in [0, 1]$

$$0 \leqslant f_n(x) = P_n f_n(x) = Q_n P f_n(x) \leqslant \max_{x \in [0, 1]} P f_n(x) \leqslant \frac{1}{\lambda} f_n(0) + K$$

In particular,

$$f_n(0) \leqslant \frac{1}{\lambda} f_n(0) + K$$

Since $\lambda > 1$,

$$f_n(0) \leqslant \frac{\lambda K}{\lambda - 1}, \qquad \forall n$$

Since f_n is nonnegative and decreasing, it follows that

$$\bigvee_0^1 f_n = f_n(0) - f_n(1) \leqslant f_n(0) \leqslant \frac{\lambda K}{\lambda - 1}, \qquad \forall n$$

By Helly's theorem, $\{f_n\}$ is precompact in $L^1(0, 1)$. Suppose $\lim_{k \to \infty} f_{n_k} = g$ for some subsequence $\{n_k\}$ of positive integers. Then from

$$\begin{aligned}\|Pg - g\| &\leqslant \|g - f_{n_k}\| + \|f_{n_k} - P_{n_k} f_{n_k}\| \\ &\quad + \|P_{n_k} f_{n_k} - P_{n_k} g\| + \|P_{n_k} g - Pg\| \\ &\leqslant \|g - f_{n_k}\| + \|f_{n_k} - g\| + \|P_{n_k} g - Pg\| \to 0\end{aligned}$$

we have $Pg = g = f^*$. Hence $\lim_{n \to \infty} f_n = f^*$, since all convergent subsequences of f_n converge to f^*. ∎

Remark 3.1. It can be shown (see, e.g., ref. 2, 8, or 15) that for n sufficiently large, P_n has a unique fixed density in T_n. Thus any sequence of fixed densities of P_n in T_n converges to f^*.

4. THE SECOND-ORDER MARKOV FINITE APPROXIMATION

In this section we prove the convergence of the second-order Markov finite approximation method for computing the absolutely continuous invariant probability measure under a piecewise convex mapping with a strong repellor.

Divide the interval $[0, 1]$ into n equal parts $I_i = [x_{i-1}, x_i]$ with the length $h = 1/n$ as before. Then the corresponding continuous piecewise quadratic finite element space T_n is $(2n+1)$-dimensional. Its standard basis consists of *B-spline functions*

$$e_{2i}(x) = u\left(\frac{x - x_i}{h}\right), \qquad i = 0, 1,..., n$$

and

$$e_{2i-1}(x) = v\left(\frac{x - x_i}{h}\right), \qquad i = 1, 2, \ldots, n$$

where

$$u(x) = (|x| - 1)^2 \chi_{[-1,1]}(x), \qquad -\infty < x < \infty$$

$$v(x) = 2x(1 - x)\chi_{[0,1]}(x), \qquad -\infty < x < \infty$$

This basis has the property that

$$\sum_{k=0}^{2n} e_k(x) \equiv 1, \qquad x \in [0, 1]$$

Notice that the support of e_{2i} is $I_i \cup I_{i+1}$ for $i = 1, 2, \ldots, n-1$, that of e_{2i-1} is I_i for $i = 1, 2, \ldots, n$, and that of e_0 and e_{2n} is I_1 and I_n, respectively. As usual, let $f_i = h^{-1} \int_{I_i} f\, dm$ be the average value of f over I_i. Now define

$$Q_n f = f_1 e_0 + \sum_{i=1}^{n-1} \frac{f_i + f_{i+1}}{2} e_{2i} + \sum_{i=1}^{n} f_i e_{2i-1} + f_n e_{2n} \tag{4}$$

Then the sequence of second-order Markov approximations $Q_n f$ strongly converges to f for any $f \in L^1(0, 1)$.[5] Obviously $\|Q_n f\|_\infty \leqslant \|f\|_\infty$ for all $f \in L^\infty(0, 1)$.

Lemma 4.1. If $f \in L^1(0, 1)$ is decreasing, then so is $Q_n f$.

Proof. Let $0 \leqslant x \leqslant y \leqslant 1$ and $x \in I_i$. First suppose $y \in I_i$ with $1 < i < n$. Then

$$Q_n f(x) = \frac{f_{i-1} + f_i}{2} e_{2(i-1)}(x) + f_i e_{2i-1}(x) + \frac{f_i + f_{i+1}}{2} e_{2i}(x)$$

$$Q_n f(y) = \frac{f_{i-1} + f_i}{2} e_{2(i-1)}(y) + f_i e_{2i-1}(y) + \frac{f_i + f_{i+1}}{2} e_{2i}(y)$$

Since f is decreasing, $f_{i-1} \geqslant f_i \geqslant f_{i+1}$. With $e_{2(i-1)}(x) \geqslant e_{2(i-1)}(y)$, $e_{2i}(x) \leqslant e_{2i}(y)$, $e_{2i-1}(x) = 1 - e_{2(i-1)}(x) - e_{2i}(x)$, and $e_{2i-1}(y) = 1 - e_{2(i-1)}(y) - e_{2i}(y)$, we have

$$\begin{aligned} &Q_n(x) - Q_n(y) \\ &\quad = \frac{f_{i-1} + f_i}{2} [e_{2(i-1)}(x) - e_{2(i-1)}(y)] + f_i [e_{2i-1}(x) - e_{2i-1}(y)] \\ &\qquad + \frac{f_i + f_{i+1}}{2} [e_{2i}(x) - e_{2i}(y)] \end{aligned}$$

$$= \frac{f_{i-1}+f_i}{2}[e_{2(i-1)}(x)-e_{2(i-1)}(y)] - f_i[e_{2(i-1)}(x)-e_{2(i-1)}(y)]$$

$$- f_i[e_{2i}(x)-e_{2i}(y)] + \frac{f_i+f_{i+1}}{2}[e_{2i}(x)-e_{2i}(y)]$$

$$= \frac{f_{i-1}-f_i}{2}[e_{2(i-1)}(x)-e_{2(i-1)}(y)] - \frac{f_i-f_{i+1}}{2}[e_{2i}(x)-e_{2i}(y)] \geqslant 0$$

Now suppose $y \in I_j$ with $1 < i < j < n$. Then

$$Q_n f(y) = \frac{f_{j-1}+f_j}{2} e_{2(j-1)}(y) + f_j e_{2j-1}(y) + \frac{f_j+f_{j+1}}{2} e_{2j}(y)$$

Since f is decreasing, we have

$$\frac{f_{i-1}+f_i}{2} \geqslant f_i \geqslant \frac{f_i+f_{i+1}}{2} \geqslant \frac{f_{j-1}+f_j}{2} \geqslant f_j \geqslant \frac{f_j+f_{j+1}}{2}$$

Now from $e_{2(i-1)}(x)+e_{2i-1}(x)+e_{2i}(x)=1$ and $e_{2(j-1)}(y)+e_{2j-1}(y)+e_{2j}(y)=1$, we obtain

$$Q_n f(x) \geqslant \frac{f_i+f_{i+1}}{2} \geqslant \frac{f_{j-1}+f_j}{2} \geqslant Q_n f(y)$$

For all other cases, the proof is similar. ∎

Let $P_n = Q_n P|_{T_n}$. Then the Markov operator P_n: $T_n \to T_n$ has a fixed density f_n.[5] As in the linear approximation case, f_n can actually be decreasing.

Lemma 4.2. P_n: $T_n \to T_n$ has a continuous piecewise-quadratic fixed density function $f_n \in D_n$ where

$$D_n = \{f \in T_n \mid f \geqslant 0, \|f\| = 1, f \text{ is decreasing}\}$$

The convergence of the piecewise quadratic Markov approximation method is exactly the same as in the previous section.

Theorem 4.1. Suppose S: $[0, 1] \to [0, 1]$ satisfies the conditions of Theorem 2.1. Let $f_n \in D_n$ be a sequence of continuous decreasing piecewise-quadratic fixed-density functions of P_n. Then f_n converges to the unique fixed density f^* of P in $L^1(0, 1)$.

5. NUMERICAL RESULTS

In this section, we present numerical results for computing the absolutely continuous invariant measure for the piecewise convex maps

$$S_1(x) = \begin{cases} x(x+3/2), & 0 \leqslant x < 1/2 \\ x(x-1/2), & 1/2 \leqslant x \leqslant 1 \end{cases}$$

and

$$S_2(x) = \begin{cases} 2x, & 0 \leqslant x < 1/2 \\ (2x-1)/(3-2x), & 1/2 \leqslant x \leqslant 1 \end{cases}$$

Since S_1 and S_2 satisfy the condition of the Lasota–Yorke theorem, there is a unique absolutely continuous invariant probability measure for them, and its density is a decreasing function. Here we use Ulam's piecewise constant approximation method and our piecewise linear method and piecewise quadratic method to compute the invariant density.

The computation was performed on the Honeywell CP6 mainframe at the University of Southern Mississippi. Double precision was used. In the algorithm, the interval $[0, 1]$ was divided into $n = 2^k$ equal subintervals with $k = 2, 3, \dots, 8$. The main numerical work in the algorithm is the evaluation of the matrix $\bar{P}_n$ of the finite-dimensional Markov operator P_n whose entries can be calculated exactly through the integration of basis functions on the inverse image of each subinterval under S, and the computation of the fixed point of $\bar{P}_n^T$, which was carried with the QR decomposition from the LINPACK subroutine dqrdc. Let f_n^0, f_n^1, and f_n^2 be the piecewise constant, piecewise linear, and piecewise quadratic approximate fixed densities of P, respectively, corresponding to the partition of $n = 2^k$. For $i = 0, 1, 2$, we used

$$e_n^i \equiv \| f_{2n}^i - f_n^i \| = \int_0^1 | f_{2n}^i(x) - f_n^i(x) | \, dx$$

to estimate the L^1-norm error of f_n^i to approximate the exact fixed density f^* of P.

In Tables I and II the first column is the number of subintervals in the partition, and the remaining ones give the corresponding errors for all the three methods. The asterisk means that the computation was not actually performed due to the storage limit. It is clear from the tables that the convergence rate is quite consistent with the order of the method.

In Tables III and IV we list approximate function values of the piecewise constant, piecewise linear, and piecewise quadratic approximate

Table I. Error Estimates for S_1

Number of subintervals	Errors: Piecewise constant	Piecewise linear	Piecewise quadratic
4	1.2926×10^{-1}	1.1913×10^{-1}	1.0608×10^{-1}
8	9.6430×10^{-2}	6.5875×10^{-2}	5.6216×10^{-2}
16	4.6338×10^{-2}	3.3130×10^{-2}	2.7132×10^{-2}
32	2.7626×10^{-2}	1.5246×10^{-2}	1.2027×10^{-2}
64	1.3738×10^{-2}	6.4226×10^{-3}	4.8539×10^{-3}
128	6.4897×10^{-3}	2.7829×10^{-3}	2.0069×10^{-3}
256	3.2951×10^{-3}	1.3837×10^{-3}	*

Table II. Error Estimates for S_2

Number of subintervals	Errors: Piecewise constant	Piecewise linear	Piecewise quadratic
4	5.5704×10^{-2}	5.4059×10^{-2}	4.3619×10^{-2}
8	3.1276×10^{-2}	2.3451×10^{-2}	1.8437×10^{-2}
16	1.6076×10^{-2}	9.1142×10^{-3}	6.8903×10^{-3}
32	8.0362×10^{-3}	3.2445×10^{-3}	2.3790×10^{-3}
64	3.7735×10^{-3}	1.0641×10^{-3}	7.6893×10^{-4}
128	1.9540×10^{-3}	3.3473×10^{-4}	2.3797×10^{-4}
256	9.6623×10^{-4}	1.0092×10^{-4}	*

Table III. Function Values at the Nodes for S_1

$\{f^0_{16}(x_i)\}_{i=1}^{16}$	$\{f^1_{16}(x_i)\}_{i=0}^{16}$	$\{f^2_{16}(x_i)\}_{i=0}^{16}$
2.722, 2.154, 1.758, 1.462, 1.303, 1.073, 1.008, 0.888, 0.623, 0.506, 0.486, 0.453, 0.435, 0.403, 0.366, 0.359	2.731, 2.443, 1.967, 1.634, 1.387, 1.199, 1.049, 0.932, 0.739, 0.571, 0.521, 0.478, 0.441, 0.409, 0.373, 0.334, 0.314	2.753, 2.463, 1.972, 1.628, 1.380, 1.194, 1.046, 0.927, 0.734, 0.566, 0.518, 0.475, 0.439, 0.407, 0.376, 0.339, 0.316

Table IV. Function Values at the Nodes for S_2

$\{f^0_{16}(x_i)\}_{i=1}^{16}$	$\{f^1_{16}(x_i)\}_{i=0}^{16}$	$\{f^2_{16}(x_i)\}_{i=0}^{16}$
1.638, 1.526, 1.363, 1.288, 1.149, 1.085, 1.002, 0.967, 0.870, 0.843, 0.784, 0.767, 0.708, 0.693, 0.663, 0.652	1.662, 1.587, 1.447, 1.325, 1.218, 1.126, 1.047, 0.978, 0.918, 0.865, 0.817, 0.774, 0.736, 0.702, 0.671, 0.643, 0.629	1.670, 1.5930, 1.452, 1.326, 1.218, 1.126, 1.046, 0.977, 0.916, 0.863, 0.815, 0.773, 0.734, 0.700, 0.669, 0.641, 0.627

fixed densities corresponding to the partition of $[0, 1]$ with $n = 16$. These tables show that all such functions are decreasing, which is guaranteed from our theoretical results.

6. CONCLUSIONS

Using a key inequality in the proof of the existence of the absolutely continuous invariant probability measure for a class of piecewise convex transformations with a strong repellor and a structure-preserving property of a class of Markov finite approximations, we proved the convergence of piecewise linear and piecewise quadratic Markov methods. Since the Markov approximations also leave the set of increasing functions invariant, as is easily seen in the proofs of Lemmas 3.1 and 4.1, for a class of non-singuar transformations S where the corresponding Frobenius–Perron operators P map increasing densities to increasing ones and satisfy a similar inequality to (2), our methods still converge.

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